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## Lecture 3

1. Define expected values
2. Properties of expected values
3. Unbiasedness of the sample mean
4. Define variances
5. Define the standard deviation
6. Calculate Bernoulli variance

## Expected values

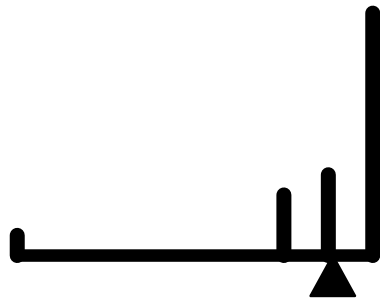
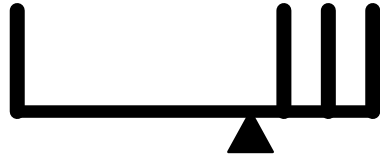
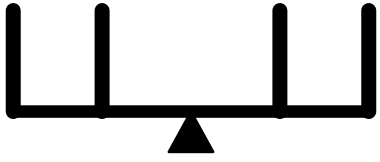
The **expected value** or **mean** of a random variable is the center of its distribution

For discrete random variable  $X$  with PMF  $p(x)$ , it is defined as follows

$$E[X] = \sum_x xp(x).$$

where the sum is taken over the possible values of  $x$

$E[X]$  represents the center of mass of a collection of locations and weights,  $\{x, p(x)\}$



## Example

- Suppose a coin is flipped and  $X$  is declared 0 or 1 corresponding to a head or a tail, respectively
- What is the expected value of  $X$ ?

$$E[X] = .5 \times 0 + .5 \times 1 = .5.$$

## Example

Suppose that a die is tossed and  $X$  is the number face up

What is the expected value of  $X$ ?

$$E[X] = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = 3.5.$$

# Continuous random variables

- For a continuous random variable,  $X$ , with density,  $f$ , the expected value is defined as follows

$$E[X] = \int_{-\infty}^{\infty} t f(t) dt.$$

- This definition borrows from the definition of center of mass for a continuous body

## Example

- Consider a density where  $f(x) = 1$  for  $x$  between zero and one
- (Is this a valid density?)
- Suppose that  $X$  follows this density; what is its expected value?

$$E[X] = \int_0^1 x dx = \frac{x^2}{2} \Big|_0^1 = 1/2.$$

# Rules about expected values

- The expected value is a linear operator
- If  $a$  and  $b$  are not random and  $X$  and  $Y$  are two random variables then
  - ▶  $E[aX + b] = aE[X] + b$
  - ▶  $E[X + Y] = E[X] + E[Y]$
- *In general* if  $g$  is a function that is not linear,

$$E[g(X)] \neq g(E[X])$$

- For example, in general,  $E[X^2] \neq E[X]^2$

## Example

You flip a coin,  $X$  and simulate a uniform random number  $Y$ , what is the expected value of their sum?

$$E[X + Y] = E[X] + E[Y] = .5 + .5 = 1$$

Another example, you roll a coin twice. What is the expected value of the average?

Let  $X_1$  and  $X_2$  be the results of the two rolls

$$E[(X_1 + X_2)/2] = \frac{1}{2}(E[X_1] + E[X_2]) = \frac{1}{2}(3.5 + 3.5) = 3.5$$

## Example

1. Let  $X_i$  for  $i = 1, \dots, n$  be a collection of random variables, each from a distribution with mean  $\mu$
2. Calculate the expected value of the sample average of the  $X_i$ .

$$\begin{aligned} E \left[ \frac{1}{n} \sum_{i=1}^n X_i \right] &= \frac{1}{n} E \left[ \sum_{i=1}^n X_i \right] \\ &= \frac{1}{n} \sum_{i=1}^n E [X_i] \\ &= \frac{1}{n} \sum_{i=1}^n \mu = \mu. \end{aligned}$$

## Remark

Therefore, the expected value of the **sample mean** is the **population mean** that it's trying to estimate

When the expected value of an estimator is what its trying to estimate, we say that the estimator is **unbiased**

# The variance

- The variance of a random variable is a measure of *spread*.
- If  $X$  is a random variable with mean  $\mu$ , the variance of  $X$  is defined as

$$\text{Var}(X) = E[(X - \mu)^2].$$

the expected (squared) distance from the mean

- Densities with a higher variance are more spread out than densities with a lower variance

- Convenient computational form

$$\text{Var}(X) = E[X^2] - E[X]^2.$$

- If  $a$  is constant then  $\text{Var}(aX) = a^2\text{Var}(X)$
- The square root of the variance is called the **standard deviation**
- The standard deviation has the same units as  $X$

## Example

What's the sample variance from the result of a toss of a die?

$$E[X] = 3.5$$

$$E[X^2] = 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 5^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6} = 15.17$$

$$\text{Var}(X) = E[X^2] - E[X]^2 \approx 2.92$$

## Example

What's the sample variance from the result of the toss of a coin with probability of heads (1) of  $p$ ?

$$E[X] = 0 \times (1 - p) + 1 \times p = p$$

$$E[X^2] = E[X] = p$$

$$\text{Var}(X) = E[X^2] - E[X]^2 = p - p^2 = p(1 - p)$$

## Example

- Suppose that a random variable is such that  $0 \leq X \leq 1$  and  $E[X] = p$
- Note  $X^2 \leq X$  so that  $E[X^2] \leq E[X] = p$
- $\text{Var}(X) = E[X^2] - E[X]^2 \leq E[X] - E[X]^2 = p(1 - p)$
- Therefore the Bernoulli variance is the largest possible for random variables bounded between 0 and 1