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Lecture 21

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- 1 Introduce Fisher's exact test
- 2 Illustrate Monte Carlo version of test

Fisher's exact test

- Fisher's exact test is “exact” because it guarantees the α rate, regardless of the sample size
- Example, chemical toxicant and 10 mice

	Tumor	None	Total
Treated	4	1	5
Control	2	3	5
Total	6	4	

- $p_1 =$ prob of a tumor for the treated mice
- $p_2 =$ prob of a tumor for the untreated mice

Continued

- $H_0 : p_1 = p_2 = p$
- Can't use Z or χ^2 because SS is small
- Don't have a specific value for p

Fisher's exact test

- Under the null hypothesis every permutation is equally likely

- observed data

```
Treatment : T T T T T C C C C C
Tumor      : T T T T N T T N N N
```

- permuted

```
Treatment : T C C T C T T C T C
Tumor      : N T T N N T T T N T
```

- Fisher's exact test uses this null distribution to test the hypothesis that $p_1 = p_2$

Hyper-geometric distribution

- X number of tumors for the treated
- Y number of tumors for the controls
- $H_0 : p_1 = p_2 = p$
- Under H_0
 - $X \sim \text{Binom}(n_1, p)$
 - $Y \sim \text{Binom}(n_2, p)$
 - $X + Y \sim \text{Binom}(n_1 + n_2, p)$

Continued

$$P(X = x \mid X + Y = z) = \frac{\binom{n_1}{x} \binom{n_2}{z-x}}{\binom{n_1+n_2}{z}}$$

This is the hypergeometric pmf

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Outline

Fisher's exact
test

The hypergeo-
metric
distribution

Fisher's exact
test in practice

Monte Carlo

$$P(X = x) = \binom{n_1}{x} p^x (1-p)^{n_1-x}$$

$$P(Y = z - x) = \binom{n_2}{z-x} p^{z-x} (1-p)^{n_2-z+x}$$

$$P(X + Y = z) = \binom{n_1 + n_2}{z} p^z (1-p)^{n_1+n_2-z}$$

Continued

$$\begin{aligned}P(X = x \mid X + Y = z) &= \frac{P(X = x, X + Y = z)}{P(X + Y = z)} \\ &= \frac{P(X = x, Y = z - x)}{P(X + Y = z)} \\ &= \frac{P(X = x)P(Y = z - x)}{P(X + Y = z)}\end{aligned}$$

Plug in and finish off yourselves

Fisher's exact test

- More tumors under the treated than the controls
- Calculate an *exact* P-value
- Use the conditional distribution = hypergeometric
- Fixes both the row and the column totals
- Yields the same test regardless of whether the rows or columns are fixed
- Hypergeometric distribution is the same as the permutation distribution given before

Tables supporting H_a

- Consider $H_a : p_1 > p_2$
- P-value requires tables as extreme or more extreme (under H_a) than the one observed
- Recall we are fixing the row and column totals
- Observed table

$$\text{Table 1} = \begin{array}{cc|c} 4 & 1 & 5 \\ 2 & 3 & 5 \\ \hline 6 & 4 & \end{array}$$

- More extreme tables in favor of the alternative

$$\text{Table 2} = \begin{array}{cc|c} 5 & 0 & 5 \\ 1 & 4 & 5 \\ \hline 6 & 4 & \end{array}$$

Calculations

$$\begin{aligned}P(\text{Table 1}) &= P(X = 4 | X + Y = 6) \\ &= \frac{\binom{5}{4} \binom{5}{2}}{\binom{10}{6}} = 0.238\end{aligned}$$

$$\begin{aligned}P(\text{Table 2}) &= P(X = 5 | X + Y = 6) \\ &= \frac{\binom{5}{5} \binom{5}{1}}{\binom{10}{6}} = 0.024\end{aligned}$$

$$P\text{-value} = 0.238 + 0.024 = 0.262$$

```
dat <- matrix(c(4, 1, 2, 3), 2)
fisher.test(dat, alternative = "greater")
```

```
-----output-----
```

```
          Fisher's Exact Test for Count Data
```

```
data:  dat
p-value = 0.2619
alt hypoth: true odds ratio is greater than 1
95 percent confidence interval:
 0.3152217      Inf
sample estimates:
odds ratio
 4.918388
```

- Two sided p-value = $2 \times$ one sided P-value
(There are other methods which we will not discuss)
- P-values are usually large for small n
- Doesn't distinguish between rows or columns
- The common value of p under the null hypothesis is called a nuisance parameter
- Conditioning on the total number of successes, $X + Y$, eliminates the nuisance parameter, p
- Fisher's exact test guarantees the type I error rate
- Exact unconditional P-value

$$\sup_p P(X/n_1 > Y/n_2; p)$$

Monte Carlo

- Observed table $X = 4$

Treatment : T T T T T C C C C C

Tumor : T T T T N T T N N N

- Permute the second row

Treatment : T T T T T C C C C C

Tumor : T N T N T T N N T T

- Simulated table $X = 3$
- Do over and over
- Calculate the proportion of tables for which the simulated $X \geq 4$
- This proportion is a Monte Carlo estimate for Fisher's exact P-value