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Section D

Handling Multiple Categorical Predictors in Multiple Linear Regression: ANOVA as a Regression Model
Sometimes regression scenarios include predictors that are not continuous, not binary, but multi-categorical.

Examples
- Subject’s race (White, African-American, Hispanic, Asian, Other)
- City of residence (Baltimore, Chicago, Tokyo, Madrid)
The Situation

• How can this type of situation be handled in a regression framework?

• We’ll explore with an example using a data set containing information about average SAT scores in 51 U.S. states (treating D.C. as a state)—the averages were based on random samples of students taken within each of the 51 states
SAT Scores Example

- The SAT (Scholastic Aptitude Test) is taken by many U.S. high school students to fulfill requirements for entry into most colleges or universities.

- The test is made up of two components: verbal and quantitative (math).

- This analysis will use the quantitative score, which ranges from 200-800 (we will refer to these simply as SAT scores for simplicity).

- This data comes from the book Statistics with Stata 8, by Lawrence Hamilton.
SAT Scores Example

- Data consists of 51 observations: the cumulative average SAT quantitative section scores for the 51 U.S. states for students taking the test in 1990

- Additional information on each observation includes geographical region of the state (West, Northeast, South, Midwest) and per-pupil education expenditures in each state in 1990
A key question
- Do average SAT scores differ across the four regions of the country and, if so, what is the magnitude of these differences?
Here is a snippet of the data in Stata (msat is mean math SAT score for the state, region is a “labeled” numerical variable)

<table>
<thead>
<tr>
<th></th>
<th>msat</th>
<th>region</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>515</td>
<td>South</td>
</tr>
<tr>
<td>2.</td>
<td>481</td>
<td>West</td>
</tr>
<tr>
<td>3.</td>
<td>490</td>
<td>West</td>
</tr>
<tr>
<td>4.</td>
<td>523</td>
<td>South</td>
</tr>
<tr>
<td>5.</td>
<td>482</td>
<td>West</td>
</tr>
<tr>
<td>6.</td>
<td>506</td>
<td>West</td>
</tr>
<tr>
<td>7.</td>
<td>468</td>
<td>N. East</td>
</tr>
</tbody>
</table>
Analysis of various testing for differences between the four states yields a p-value of less than 0.001

So there are at least some statistical differences in SAT quantitative scores across the four regions—but in order to find out which regions are statistically different and to figure out by how large (and in what direction) these differences occur would require a lot of t-tests
ANOVA as a Regression Model

- Could this analysis be done by a linear regression relating SAT scores to region?

- How can we handle a predictor that takes on four categories?
ANOVA as a Regression Model

- Arbitrarily give each region a numerical value ($x_1 = 1$ for Western region states, 2 for Northeastern states, 3 for Southern states, and 4 for mid-Western states for example) and fit SLR of

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1$$

- Where $\hat{y}$ is estimated mean SAT score, and $x_1$ is region as defined above
ANOVA as a Regression Model

- This is not a good idea!!

- Coding is arbitrary, could have assigned \( x_1 = 1 \) for Midwest, etc. . . .
  - Estimated coefficient of region will depend on arbitrary coding

- Coding “assumes” mean SAT score differences between regions is “incremental”
  - Example—difference in average SAT scores between Southern states \( (x_1 = 3) \) and Western States \( (x_1 = 1) \) is twice the difference between Northeastern States \( (x_1 = 2) \) and Western States \( (x_1 = 1) \)
ANOVA as a Regression Model

- Alternative approach—designate one region as “reference” region, say Western region, and make binary indicators for each of the three other regions
  - $x_1 = 1$ if Northeastern state, 0 otherwise
  - $x_2 = 1$ if Southern state, 0 otherwise
  - $x_3 = 1$ if mid-Western state, 0 otherwise
ANOVA as a Regression Model

- Here is a table showing the $x$ values for each region:

<table>
<thead>
<tr>
<th>Region</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>West</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Northeast</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>South</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Mid-west</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
ANOVA as a Regression Model

- Fit the regression model

\[ \hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \hat{\beta}_3 x_3 \]

- Here, each coefficient estimates mean SAT score difference between a region that has a corresponding \( x \) value of 1 and the reference region (Western states).

- Notice, the intercept has meaning here—it’s the estimated mean when all \( x \)s are 0, the estimated mean SAT score for Western states!
Example

For Northeastern states \((x_1 = 1, x_2 = 0, x_3 = 0)\) model predicts

\[
\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 \times 1 + \hat{\beta}_2 \times 0 + \hat{\beta}_3 \times 0
\]

\[
= \hat{\beta}_0 + \hat{\beta}_1
\]

For Western states \((x_1 = 0, x_2 = 0, x_3 = 0)\) model predicts

\[
\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 \times 0 + \hat{\beta}_2 \times 0 + \hat{\beta}_3 \times 0
\]

\[
= \hat{\beta}_0
\]
Example

So \( \hat{y}_{NE} - \hat{y}_W = \hat{\beta}_0 + \hat{\beta}_1 - \hat{\beta}_0 = \hat{\beta}_1 \)

Similar results can be shown for other coefficients
ANOVA as a Regression Model

- Stata results

- Notice, data in the following format . . .

<table>
<thead>
<tr>
<th></th>
<th>msat</th>
<th>region</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>515</td>
<td>3</td>
</tr>
<tr>
<td>2.</td>
<td>481</td>
<td>1</td>
</tr>
<tr>
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<td>490</td>
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<td>523</td>
<td>3</td>
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<tr>
<td>5.</td>
<td>482</td>
<td>1</td>
</tr>
<tr>
<td>6.</td>
<td>506</td>
<td>1</td>
</tr>
<tr>
<td>7.</td>
<td>468</td>
<td>2</td>
</tr>
<tr>
<td>8.</td>
<td>464</td>
<td>3</td>
</tr>
</tbody>
</table>
ANOVA as a Regression Model

- “xi” option before regression command will automatically create binary indicators for a multi-categorical variable

- Syntax
  - `xi: regress msat i.region`
ANOVA as a Regression Model

- Stata results

```
.xi: regress msat i.region
(i.region _Iregion_1-4) (naturally coded; _Iregion_1 omitted)

Source | SS       df       MS
----------+------------------------
Model     | 26433.6728  3  8811.22428
Residual | 33050.8072  46  718.495808
----------+------------------------
Total     | 59484.48   49  1213.96898

Number of obs = 50
F(3, 46) = 12.26
Prob > F = 0.0000
R-squared = 0.4444
Adj R-squared = 0.4081
Root MSE  = 26.805

msat | Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
----------+--------------------------------------------------
_Iregion_2 | -32.01709   11.62333    -2.75  0.008     -55.41364   -8.620547
_Iregion_3 | -11.64904   10.00874    -1.16  0.250     -31.79559    8.497512
_Iregion_4 |  35.45513   10.73055     3.30  0.002     13.85576   57.05451
_cons      |  498.4615     7.434306    67.05  0.000     483.4971    513.426
```

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Resulting regression equation

\[
\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \hat{\beta}_3 x_3
\]

\[
\hat{y} = 498.5 - 32.0 x_1 - 11.6 x_2 + 35.5 x_3
\]
ANOVA as a Regression Model

- Overall F-test

```
. xi: regress msat i.region
i.region   _Region_1-4           (naturally coded; _Region_1 omitted)

<table>
<thead>
<tr>
<th>Source</th>
<th>SS</th>
<th>df</th>
<th>MS</th>
<th>Number of obs = 50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>26433.6728</td>
<td>3</td>
<td>8811.22428</td>
<td>F( 3, 46) = 12.26</td>
</tr>
<tr>
<td>Residual</td>
<td>33050.8072</td>
<td>46</td>
<td>719.495808</td>
<td>Prob &gt; F = 0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>R-squared = 0.4444</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Adj R-squared = 0.4081</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Root MSE = 26.805</td>
</tr>
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<tr>
<td>Model</td>
<td>26433.6728</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Adj R-squared = 0.4081</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Root MSE = 26.805</td>
</tr>
</tbody>
</table>

| Variable | Coef.   | Std. Err. | t      | P>|t|  | 95% Conf. Interval |
|----------|---------|-----------|--------|------|-------------------|
| _Region_2 | -32.01709 | 11.62333  | -2.75  | 0.008 | -55.41364 to -8.620547 |
| _Region_3 | -11.64904 | 10.00874  | -1.16  | 0.250 | -31.79559 to 8.497512 |
| _Region_4 | 35.45513  | 10.7305   | 3.30   | 0.002 | 13.85576 to 57.0545 |
| _cons    | 498.4615  | 7.434306  | 67.05  | 0.000 | 483.4971 to 513.426 |
```
ANOVA as a Regression Model

- This is the overall test for . . .
  - $H_0$: $\beta_1 = \beta_2 = \beta_3$: no differences in mean SAT scores across the four regions
  - $H_a$: at least one region has different mean SAT scores than the others
  - This is the same exact test that we did with the traditional ANOVA approach
Some of the estimated regional differences

```
.xi: regress msat i.region
  i.region       _Region_1-4        (naturally coded; _Region_1 omitted)

Source | SS     df  MS
--------|--------|------|------
Model   | 26433.6728  3  8811.22428  \( F(3, 46) = 12.26 \)
Residual| 33050.8072  46  718.495808 \( \text{Prob} > F = 0.000 \)
--------|--------|------|------
Total   | 59484.48  49  1213.96898 \( R^2 = 0.4444 \)
        |        |      | \( \text{Adj R}^2 = 0.4081 \)
        |        |      | \( \text{Root MSE} = 26.805 \)

| msat | Coef. | Std. Err. | t    | P>|t| | [95% Conf. Interval] |
|------|-------|-----------|------|------|---------------------|
| _Region_2 | -32.01709 | 11.62333 | -2.75 | 0.009 | -55.41364 to -8.620547 |
| _Region_3 | -11.64304 | 10.00874 | -1.16 | 0.250 | -31.79559 to 8.497512 |
| Region 4  | 35.45513  | 10.7305  | 3.30  | 0.002 | 13.85576 to 57.0545  |
| cons     | 498.4615  | 7.434306 | 67.05 | 0.000 | 483.4971 to 513.426  |
```
Results

- A statistically significant relationship was found between mean SAT scores and student’s region of the country (p < .0001 by F-test)

- Students from northeastern states had SAT scores of 32 points lower on average than students from western states (95% CI 8.6 to 55.4 points lower)
Results

- Students from southern states had SAT scores of 11.6 points lower on average than students from western states (95% CI 31.6 points lower to 8.5 points higher)

- Students from mid-western states had SAT scores of 35.5 points higher on average than students from western states (95% CI 13.9 points to 57.0 points higher)

- Regional differences account for 44% of the variation in SAT scores
What about other comparisons—for example, SAT scores for Northeastern states to mid-western states?

- One approach—recode indicators for region making “mid-west” the reference group—more work!
- Another option—use existing coefficients
Recall $\hat{\beta}_1 = \hat{y}_{NE} - \hat{y}_W = -32.0$ estimates the average difference in SAT scores for northeastern states minus (compared to) western states.

Recall $\hat{\beta}_3 = \hat{y}_{MW} - \hat{y}_W = 35.5$ estimates the average difference in SAT scores for mid-western states minus (compared to) western states.
Results

- So:

\[ \hat{\beta}_1 - \hat{\beta}_3 = \hat{y}_{NE} - \hat{y}_W - (\hat{y}_{MW} - \hat{y}_W) \]

\[ = \hat{y}_{NE} - \hat{y}_W - \hat{y}_{MW} + \hat{y}_W \]

\[ = \hat{y}_{NE} - \hat{y}_{MW} \]

- So the estimated mean difference in SAT scores between northeastern states and mid-western states is given by (-32.0-35.5) = -67.5 points
Results

- We can employ Stata to do this and get a 95% CI (just FYI)

- The “lincom” command can be run after any regression to give estimates for differences in coefficients
We need to use names Stata gives to coefficients in command

```
.xi: regress msat i.region
i.region  _Iregion_1-4    (naturally coded; _Iregion_1 omitted)

Source |     SS      df    MS
---------|----------|--------|--------
Model    | 26433.6728   3  8811.22428
Residual | 33050.8072   46  718.495809
---------|----------|--------|--------
Total    | 59484.48    49 1213.96898

Number of obs = 50
F(  3,   46) = 12.26
Prob > F    = 0.0000
R-squared   = 0.4444
Adj R-squared = 0.4081
Root MSE    = 26.805

-----------------------------------------------
msat |     Coef.   Std. Err.     t   P>|t|    [95% Conf. Interval]
---------|----------|-----------------|--------|--------|-----------------------------------------------
  _Iregion_2 |  -32.01709  11.62333   -2.75  0.008  [-55.41364  -8.620547]
  _Iregion_3 |  -11.64904  10.00874   -1.16  0.250  [-31.79559   8.497512]
  _Iregion_4 |   35.45513   10.7305   3.30  0.002    13.85576   57.0545
     _cons |    498.4615   7.434306   67.05  0.000   483.4971   513.426
```
ANOVA as a Regression Model

- Syntax
  - `lincom _Iregion_2 - _Iregion_4`

```
.lincom _Iregion_2 - _Iregion_4
( 1)  _Iregion_2 - _Iregion_4  =  0

                                  |      Coef.     Std. Err.     t     P>|t|     [95% Conf. Interval]
---------------------------------|-----------------|-----------------|--------|---------|------------------|------------------|
msat |                     |                 |        |         |                  |                  |
---------------------------------|-----------------|-----------------|--------|---------|------------------|------------------|
       (1) |        -67.47222  |  11.81979       | -5.71  |  0.000  |    -91.26423    |   -43.68021      |
```

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Recap

- ANOVA is just a specific form of linear regression

- In general, if we have a categorical predictor with k categories, we designate one category as the reference group and create \( k-1 \) binary indicators \( x_1, x_2, x_{k-1} \) for all other levels of the predictor.

- Coefficients are interpretable as mean difference in the outcome between each of the \( k-1 \) categories and the reference group.
Advantages

- Not only do we get an overall test for any mean outcome differences between the groups being compared, we also get estimates and 95% CIs for some of the differences.

- This approach also gives a $R^2$ value.

- We can also expand the regression model to include more predictors (example—SAT scores predicted by both region and per-pupil state expenditures).