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# ANOVA, still

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$\{Y_{ti}\}$  independent with  $Y_{ti} \sim \text{normal}(\mu_t, \sigma)$  for  $t = 1 \dots k$ .

Test  $H_0 : \mu_1 = \mu_2 = \dots = \mu_k$

The usual statistic:

$$F = M_B/M_W = \frac{\sum_t n_t (\bar{Y}_t - \bar{Y}_{..})^2/k}{\sum_t \sum_i (Y_{ti} - \bar{Y}_t)^2 / (\sum_t n_t - k)}$$

**P-values:** (a) Use the  $F(k, \sum n_t - k)$  distribution.  
(b) Use a permutation test.

**Assumptions:** (a) Underlying dist'ns are normal with common SD.  
(b) Underlying dist'ns are the same.

## Non-parametric ANOVA

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An alternative approach: the Kruskal-Wallis test.

Rank all of the observations from 1, 2, ..., N.

Let  $R_{ti}$  = the rank for observation  $Y_{ti}$ .

Let  $\bar{R}_t = \sum_i R_{ti}/n_t$  = the average rank for group t.

Null hypothesis,  $H_0$ : the underlying distributions are all the same.

$$E(\bar{R}_t \mid H_0) = \frac{N+1}{2}$$

$$SD(\bar{R}_t \mid H_0) = \sqrt{\frac{(N+1)(N-n_t)}{12 n_t}}$$

## Kruskal-Wallis test statistic

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$$H = \sum_t \left( \frac{N - n_t}{N} \right) \times \left[ \frac{\bar{R}_{t.} - E(\bar{R}_{t.} | H_0)}{SD(\bar{R}_{t.} | H_0)} \right]^2$$
$$= \dots = \frac{12}{N(N+1)} \sum_t n_t \left[ \bar{R}_{t.} - \left( \frac{N+1}{2} \right) \right]^2$$

Under  $H_0$ , and if the sample sizes are large,  $H \sim \chi^2(df = k - 1)$ .

Alternatively, we could use a **permutation test** to estimate a P-value.

The function `kruskal.test()` in R will calculate the statistic.

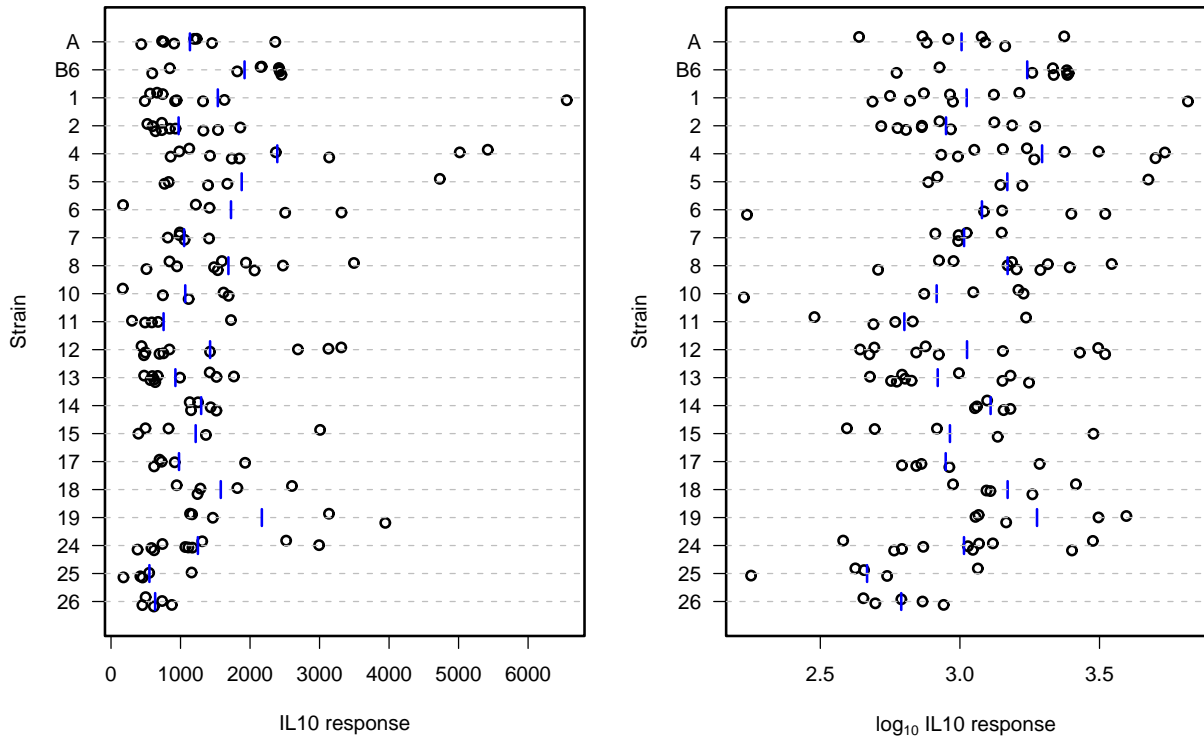
## Note

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In the case of two groups, the Kruskal-Wallis test reduces exactly to the **Wilcoxon rank-sum test**.

This is just like how **ANOVA** is equivalent to the **two-sample t test**.

# Example



## ANOVA Tables

Original scale / 1000:

source	SS	df	MS	F	P-value
between strains	33	20	1.69	1.70	0.042
within strains	124	125	0.99		
total	157	145			

permutation P-value = 0.043

log<sub>10</sub> scale:

source	SS	df	MS	F	P
between strains	3.35	20	0.167	2.25	0.0036
within strains	9.29	125	0.074		
total	12.63	145			

permutation P-value = 0.003

# K-W results

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The observed Kruskal-Wallis statistic for these data was **41.32**. (Note that it doesn't matter whether you take logs.)

Since there were 21 strains, we can compare this to a  $\chi^2$  distribution with **20** degrees of freedom. Thus we obtain the **P-value = 0.003**.

With a permutation test, I got  $\hat{P} = 0.0015$  (on the basis of 10,000 simulations).

## In the case of ties...

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In the case of ties, we assign the **average rank** to each.

Example:

<b>A:</b>	3.5	3.7	4.0	4.2	4.3				
<b>B:</b>				3.9			4.3	4.5	
<b>C:</b>	3.1	3.6			4.0		4.3		
	(1)	(2)	(3)	(4)	(5)	(6/7)	(8)	(9/10/11)	(12)
						↓		↓	
						6.5		10	

Then we apply a correction factor.

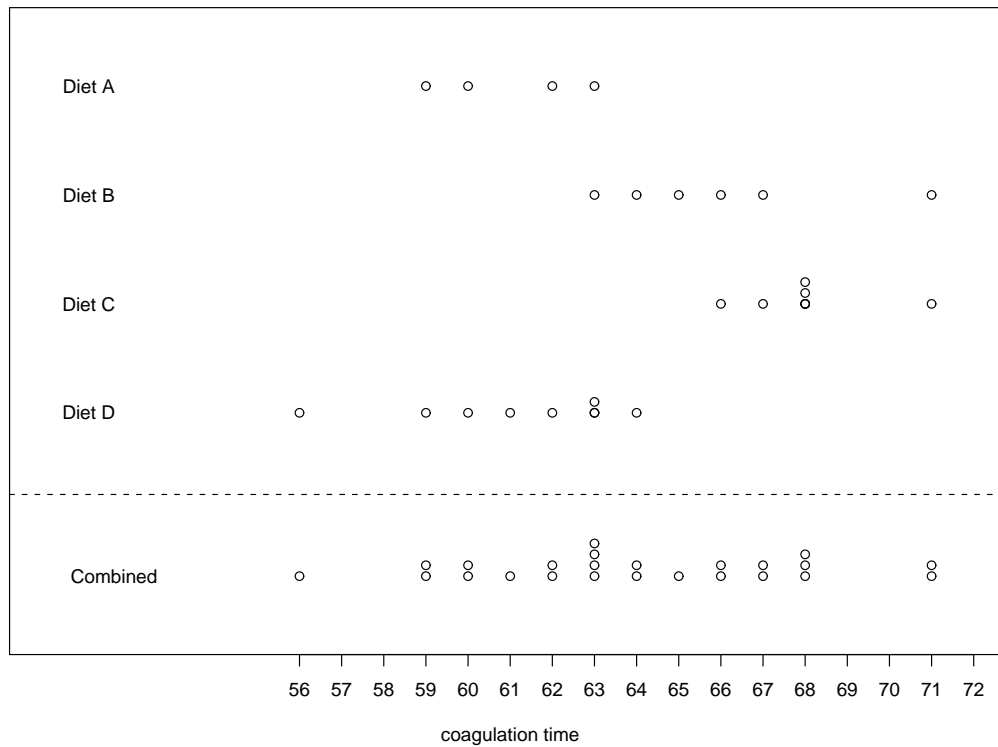
Let  $N = \sum_t n_t$  and  $T_i =$  no. observations in the  $i$ th set of ties (can be 1).

$$\text{Let } D = 1 - \frac{\sum_i (T_i^3 - T_i)}{N^3 - N}$$

Use the statistic  $H' = H/D$ .

Note that  $D \leq 1$  and so  $H' \geq H$ . For the example,  $D = 1 - \frac{(2^3-2)+(3^3-3)}{12^3-12} \approx 0.983$ .

# Blood coagulation time



## Example (continued)

A	B	C	D	rank	avg rank
			56	1	1
59				2	2.5
		59		3	2.5
60				4	4.5
		60		5	4.5
		61		6	6
62				7	7.5
		62		8	7.5
63				9	10.5
	63			10	10.5
		63		11	10.5
		63		12	10.5
64				13	13.5
		64		14	13.5
65				15	15
66				16	16.5
	66			17	16.5
67				18	18.5
	67			19	18.5
68				20	21
	68			21	21
	68			22	21
71				23	23.5
	71			24	23.5

## Example (continued)

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<b>A</b>	62	60	63	59					61
	7.5	4.5	10.5	2.5					6.25
<b>B</b>	63	67	71	64	65	66			66
	10.5	18.5	23.5	13.5	15.0	16.5			16.25
<b>C</b>	68	66	71	67	68	68			68
	21.0	16.5	23.5	18.5	21.0	21.0			20.25
<b>D</b>	56	62	60	61	63	64	63	59	61
	1.0	7.5	4.5	6.0	10.5	13.5	10.5	2.5	7.00

## Calculation of K-W test statistic

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	A	B	C	D	
$n_t$	4	6	6	8	$N = 24$
$\bar{R}_t$	6.25	16.25	20.25	7.00	$\frac{N+1}{2} = 12.5$

$$\begin{aligned}
 H &= \frac{12}{N(N+1)} \sum_t n_t \left[ \bar{R}_t - \left( \frac{N+1}{2} \right) \right]^2 \\
 &= \frac{12}{24 \times 25} \left\{ 4 \times (6.25 - 12.5)^2 + \dots + 8 \times (7.00 - 12.5)^2 \right\} \\
 &= 16.86
 \end{aligned}$$

The ties:  $T_i = ( 1 \ 2 \ 2 \ 1 \ 2 \ 4 \ 2 \ 1 \ 2 \ 2 \ 3 \ 2 )$

$$D = 1 - \sum_i (T_i^3 - T_i) / (N^3 - N) = \dots = 0.991$$

$$H' = H/D = 16.86 / 0.991 = 17.02 \quad [df = 3] \quad P\text{-value} \approx 0.0007$$

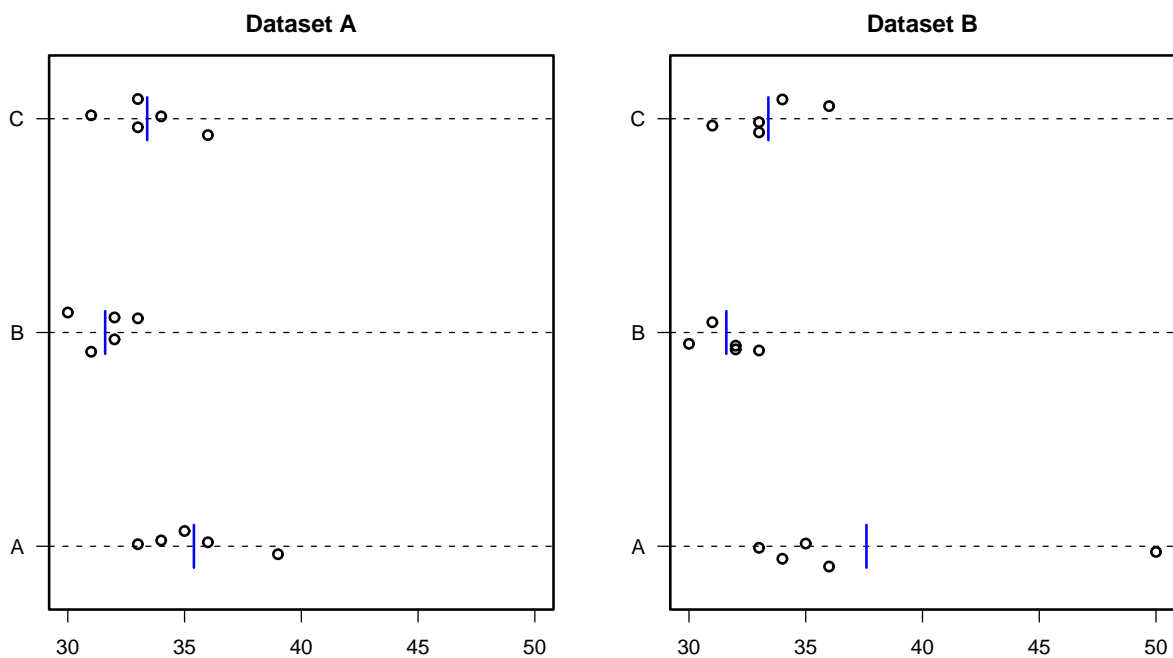
# A few points

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- **Calculation of P-values:** (avoiding type I errors)
  - F statistic: F distribution (requires normality)
  - K-W statistic:  $\chi^2$  distribution (requires large samples)
  - Either statistic: Permutation tests
- **Power:** (avoiding type II errors)
  - K-W statistic more resistant to outliers
  - F statistic more powerful in the case of normality
- K-W statistic: don't need to worry about transformations.

# A fake example

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# Results

Dataset	Method	Statistic	nominal P-value	Permu'n P-value
<b>A</b>	ANOVA	5.48	0.020	0.017
	K-W	7.64	0.022	0.012
<b>B</b>	ANOVA	2.64	0.112	0.023
	K-W	7.64	0.022	0.012

## Distributions

