Comparing Means among Two (or More) Independent Populations

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Lecture Topics

- CIs for mean difference between two independent populations
- Two sample t-test
- Non-parametric alternative, Mann Whitney (FYI, optional)
- Comparing means amongst more than two independent populations: ANOVA
Section A

Two Sample t-test: The Resulting Confidence Interval
Comparing Two Independent Groups

- “A Low Carbohydrate as Compared with a Low Fat Diet in Severe Obesity”*
  - 132 severely obese subjects randomized to one of two diet groups
  - Subjects followed for a six month period

- At the end of study period
  - “Subjects on the low-carbohydrate diet lost more weight than those on a low fat diet (95% confidence interval for the difference in weight loss between groups, -1.6 to -6.2 kg; p < .01)”

**Comparing Two Independent Groups: Diet Types Study**

- **Scientific question**
  - Is weight change associated with diet type?

<table>
<thead>
<tr>
<th>Diet Group</th>
<th>Low-Carb</th>
<th>Low-Fat</th>
</tr>
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<tbody>
<tr>
<td>Number of subjects (n)</td>
<td>64</td>
<td>68</td>
</tr>
<tr>
<td>Mean weight change (kg)</td>
<td>-5.7</td>
<td>-1.8</td>
</tr>
<tr>
<td>Post-diet less pre-diet</td>
<td></td>
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</tr>
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<td>Standard deviation of weight changes (kg)</td>
<td>8.6</td>
<td>3.9</td>
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Diet Type and Weight Change

- 95% CIs for weight change by diet group

- Low Carb: $-5.7 \pm 2 \times \frac{8.6}{\sqrt{64}} \rightarrow -5.7 \pm 2 \times 1.08 \approx (-7.8 \text{ kg}, -3.5 \text{ kg})$

- Low Fat: $-1.8 \pm 2 \times \frac{3.9}{\sqrt{68}} \rightarrow -1.8 \pm 2 \times .47 \approx (-2.7 \text{ kg}, -0.9 \text{ kg})$
In statistical terms, is there a non-zero difference in the average weight change for the subjects on the low-fat diet as compared to subjects on the low-carbohydrate diet?

- 95% CIs for each diet group mean weight change do not overlap, but how do you quantify for the difference?

The comparison of interest is not “paired”

- There are different subjects in each diet group

For each subject a change in weight (after diet—before weight) was computed

- However, the authors compared the changes in weight between two independent groups!
Comparing Two Independent Groups

- How do we calculate
  - Confidence interval for difference?
  - p-value to determine if the difference in two groups is “significant?”

- Since we have large samples (both greater than 60) we know the sampling distributions of the sample means in both groups are approximately normal

- It turns out the difference of quantities, which are (approximately) normally distributed, are also normally distributed
So, the big news is . . .

- The sampling distribution of the difference of two sample means, each based on large samples, approximates a normal distribution
- This sampling distribution is centered at the true mean difference, $\mu_1 - \mu_2$
Simulated sampling distribution of sample mean weight change: low carbohydrate diet group
Simulated sampling distribution of sample mean weight change: low fat diet group
Simulated Sampling Dist’n of Sample Mean Weight Loss

- Simulated sampling distribution of sample mean weight change: low fat diet group

Simulated Sampling Distribution: Diff in Mean Weight Change
1,000 Random Samples: Low Carb (n=64) - Low Fat (n=68)
Simulated Sampling Dist’n of Sample Mean Weight Loss

- Side by side boxplots
95% Confidence Interval for Difference in Means

- Our most general formula

\[ \text{best estimate from sample} \pm 2 \times SE(\text{best estimate from sample}) \]

- The best estimate of a population mean difference based on sample means:

\[ \bar{x}_1 - \bar{x}_2 \]

- Here, \( \bar{x}_1 \) may represent the sample mean weight loss for the 64 subjects on the low carbohydrate diet, and \( \bar{x}_2 \) the mean weight loss for the 68 subjects on the low fat diet.
So, $\bar{x}_1 - \bar{x}_2 = -5.7 - (-1.8) = -3.9 \text{ kg}$: hence the formula for the 95% CI for $\mu_1 - \mu_2$ is:

$$-3.9 \pm 2 \times SE(\bar{x}_1 - \bar{x}_2)$$

Where $SE(\bar{x}_1 - \bar{x}_2)$ = standard error of the difference of two sample means
Two Independent (Unpaired) Groups

- The standard error of the difference for two independent samples is calculated differently than we did for paired designs
  - With paired design we reduced data on two samples to one set of differences between two groups

- Statisticians have developed formulas for the standard error of the difference

- These formulas depend on sample sizes in both groups and standard deviations in both groups

- The $SE(\bar{x}_1 - \bar{x}_2)$ is greater than either $SE(\bar{x}_1)$ or $SE(\bar{x}_2)$
  - Why do you think this is?
- Variation from independent sources can be added
  Why do you think this is additive

\[
SE (\bar{x}_1 - \bar{x}_2) = \sqrt{\frac{\sigma^2_1}{n_1} + \frac{\sigma^2_2}{n_2}}
\]

Of course, we don’t know \( \sigma_1 \) and \( \sigma_2 \): so we estimate with \( s_1 \) and \( s_2 \) to get an estimated standard error:

\[
\hat{SE} (\bar{x}_1 - \bar{x}_2) = \sqrt{\frac{s^2_1}{n_1} + \frac{s^2_2}{n_2}} = \sqrt{\hat{SE}(\bar{x}_1)^2 + \hat{SE}(\bar{x}_2)^2}
\]
Comparing Two Independent Groups: Diet Types Study

- Recall the data from the weight change/diet type study

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\[
S\hat{E} (\bar{X}_1 - \bar{X}_2) = \sqrt{\frac{8.6^2}{64} + \frac{3.9^2}{68}} \approx 1.17
\]
So in this example, the estimated 95% for the true mean difference in weight between the low-carbohydrate and low-fat diet groups is:

\[-3.9 \pm 2 \times \hat{SE}(\bar{x}_1 - \bar{x}_2)\]

\[-3.9 \pm 2 \times 1.17\]

\[-3.9 \pm 2 \times 1.17\]

\[-6.24 \text{ kg to } -1.56 \text{ kg} \approx\]

\[-6.2 \text{ kg to } -1.6 \text{ kg}\]
“Subjects on the low-carbohydrate diet lost more weight than those on a low fat diet (95% confidence interval for the difference in weight loss between groups, -1.6 to -6.2 kg; p< .01)”

So those on the low carb diet lost more on average by 3.9 kg: after accounting for sampling variability this excess average loss over the low-fat diet group could be as small as 1.6 kg or as large as 6.2 kg.

This confidence interval does not include 0, suggesting a real population level association between type of diet (low-carb or low-fat) and weight loss.