Example


Place tick on clay island surrounded by water, with two capillary tubes: one treated with deer-gland-substance; one untreated.

Does the tick go to the treated or the untreated tube?

<table>
<thead>
<tr>
<th>Tick sex</th>
<th>Leg</th>
<th>Deer sex</th>
<th>treated</th>
<th>untreated</th>
</tr>
</thead>
<tbody>
<tr>
<td>male</td>
<td>fore</td>
<td>female</td>
<td>24</td>
<td>5</td>
</tr>
<tr>
<td>female</td>
<td>fore</td>
<td>female</td>
<td>18</td>
<td>5</td>
</tr>
<tr>
<td>male</td>
<td>fore</td>
<td>male</td>
<td>23</td>
<td>4</td>
</tr>
<tr>
<td>female</td>
<td>fore</td>
<td>male</td>
<td>20</td>
<td>4</td>
</tr>
<tr>
<td>male</td>
<td>hind</td>
<td>female</td>
<td>17</td>
<td>8</td>
</tr>
<tr>
<td>female</td>
<td>hind</td>
<td>female</td>
<td>25</td>
<td>3</td>
</tr>
<tr>
<td>male</td>
<td>hind</td>
<td>male</td>
<td>21</td>
<td>6</td>
</tr>
<tr>
<td>female</td>
<td>hind</td>
<td>male</td>
<td>25</td>
<td>2</td>
</tr>
</tbody>
</table>

Is the tick more likely to go to the treated tube?

Test for a proportion

Suppose $X \sim \text{binomial}(n, p)$.

Test $H_0 : p = \frac{1}{2}$ vs $H_a : p \neq \frac{1}{2}$

Reject $H_0$ if $X \geq H$ or $X \leq L$

Choose $H$ and $L$ such that

$$\Pr(X \geq H \mid p = \frac{1}{2}) \leq \alpha/2 \text{ and } \Pr(X \leq L \mid p = \frac{1}{2}) \leq \alpha/2$$

Thus $\Pr(\text{Reject } H_0 \mid H_0 \text{ is true}) \leq \alpha$.

The difficulty: The binomial distribution is hard to work with. Because of its discrete nature, you can’t get exactly your desired significance level ($\alpha$).
Consider $X \sim \text{binomial}(n=29, p)$

Test of $H_0: p = \frac{1}{2}$ vs $H_a: p \neq \frac{1}{2}$ at significance level $\alpha = 0.05$

Lower critical value:

$q\text{binom}(0.025, 29, 0.5) = 9$

$\Pr(X \leq 9) = p\text{binom}(9, 29, 0.5) = 0.031 \rightarrow L = 8$

Upper critical value:

$q\text{binom}(0.975, 29, 0.5) = 20$

$\Pr(X \geq 20) = 1 - p\text{binom}(20,29,0.5) = 0.031 \rightarrow H = 21$

Reject $H_0$ if $X \leq 8$ or $X \geq 21$. (For testing $H_0: p = \frac{1}{2}$, $H = n - L$.)
Consider $X \sim \text{binomial}(n=29, p)$

Test of $H_0 : p = \frac{1}{2}$ vs $H_a : p \neq \frac{1}{2}$ at significance level $\alpha = 0.05$

Reject $H_0$ if $X \leq 8$ or $X \geq 21$.

Actual significance level:

\[
\alpha = \Pr(X \leq 8 \text{ or } X \geq 21 \mid p = \frac{1}{2})
= \Pr(X \leq 8 \mid p = \frac{1}{2}) + [1 - \Pr(X \leq 20 \mid p = \frac{1}{2})]
= \text{pbinom}(8, 29, 0.5) + 1 - \text{pbinom}(20, 29, 0.5)
= 0.024
\]

If we used, instead, “Reject $H_0$ if $X \leq 9$ or $X \leq 20$,” the significance level would be:

\[
\text{pbinom}(9, 29, 0.5) + 1 - \text{pbinom}(19, 29, 0.5) = 0.061
\]

Example

Observe $X = 24$ (for $n = 29$)

Reject $H_0 : p = \frac{1}{2}$ if $X \leq 8$ or $X \geq 21$.

Thus we reject $H_0$ and conclude that the ticks were more likely to go to the deer-gland-substance-treated tube.

\[
P\text{-value} = 2 \times \Pr(X \geq 24 \mid p = \frac{1}{2})
= 2 \times (1 - \text{pbinom}(23, 29, 0.5))
= 5/10,000.
\]

Alternatively: \texttt{binom.test(24, 29)}
Example 2

Observe $X = 17$ (for $n = 25$); assume $X \sim \text{binomial}(n=25, \ p)$

Test $H_0 : p = \frac{1}{2}$ vs $H_a : p \neq \frac{1}{2}$

Rejection rule: Reject $H_0$ if $X \leq 7$ or $X \geq 18$

$q\text{binom}(0.025, 25, 0.5) = 8$
$p\text{binom}(8, 25, 0.5) = 0.054$
$p\text{binom}(7, 25, 0.5) = 0.022$

Significance level:

$p\text{binom}(7, 25, 0.5) + 1 - p\text{binom}(17, 25, 0.5) = 0.043$

Since we observed $X = 17$, we fail to reject $H_0$

$P\text{-value} = 2 \times (1 - p\text{binom}(16, 25, 0.5)) = 0.11$

Confidence interval for a proportion

Suppose $X \sim \text{binomial}(n=29, \ p)$ and we observe $X = 24$.

Consider the test of $H_0 : p = p_0$ vs $H_a : p \neq p_0$

We reject $H_0$ if

$\Pr(X \leq 24 \mid p = p_0) \leq \alpha/2 \quad \text{or} \quad \Pr(X \geq 24 \mid p = p_0) \leq \alpha/2$

95% confidence interval for $p$:

The set of $p_0$ for which a two-tailed test of $H_0 : p = p_0$ would not be rejected, for the observed data, with $\alpha = 0.05$.

The “plausible” values of $p$. 
Example

$X \sim \text{binomial}(n=29, \ p)$; observe $X = 24$

Lower bound of 95% confidence interval:
Largest $p_0$ such that $\Pr(X \geq 24 \mid p = p_0) \leq 0.025$

Upper bound of 95% confidence interval:
Smallest $p_0$ such that $\Pr(X \leq 24 \mid p = p_0) \leq 0.025$

In R: \hspace{1em} \text{binom.test}(24, 29)

95% CI for $p$: (0.642, 0.942)

Note: $\hat{p} = 24/29 = 0.83$ is not the midpoint of the CI

\begin{itemize}
\item Binomial(n=29, p=0.64)
\item Binomial(n=29, p=0.94)
\end{itemize}
Example 2

$X \sim \text{binomial}(n=25, p)$; observe $X = 17$

Lower bound of 95% confidence interval:

$p_L$ such that 17 is the 97.5 percentile of $\text{binomial}(n=25, p_L)$

Upper bound of 95% confidence interval:

$p_H$ such that 17 is the 2.5 percentile of $\text{binomial}(n=25, p_H)$

In R: \[ \text{binom.test}(17, 25) \]

95% CI for $p$: (0.465, 0.851)

Again, $\hat{p} = 17/25 = 0.68$ is not the midpoint of the CI
The case $X = 0$

Suppose $X \sim \text{binomial}(n, p)$ and we observe $X = 0$.

**Lower limit of 95% confidence interval for** $p$: 0

**Upper limit of 95% confidence interval for** $p$:

$p_H$ such that

$$\Pr(X \leq 0 \mid p = p_H) = 0.025$$

$$\implies \Pr(X = 0 \mid p = p_H) = 0.025$$

$$\implies (1 - p_H)^n = 0.025$$

$$\implies 1 - p_H = \sqrt[n]{0.025}$$

$$\implies p_H = 1 - \sqrt[n]{0.025}$$

In the case $n = 10$ and $X = 0$, the 95% CI for $p$ is $(0, 0.31)$

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**A mad cow example**

New York Times, Feb 3, 2004:

The department [of Agriculture] has not changed last year’s plans to test 40,000 cows nationwide this year, out of 30 million slaughtered. Janet Riley, a spokeswoman for the American Meat Institute, which represents slaughterhouses, called that “plenty sufficient from a statistical standpoint.”

Suppose that the 40,000 cows tested are chosen at random from the population of 30 million cows, and suppose that 0 (or 1, or 2) are found to be infected.

<table>
<thead>
<tr>
<th>No. infected</th>
<th>Obs'd</th>
<th>Est'd</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0 – 2763</td>
</tr>
<tr>
<td>1</td>
<td>750</td>
<td>19</td>
<td>19 – 4173</td>
</tr>
<tr>
<td>2</td>
<td>1500</td>
<td>181</td>
<td>181 – 5411</td>
</tr>
</tbody>
</table>

How many of the 30 million total cows would we estimate to be infected?

What is the 95% confidence interval for the total number of infected cows?
The case $X = n$

Suppose $X \sim \text{binomial}(n, p)$ and we observe $X = n$.

Upper limit of 95% confidence interval for $p$: 1

Lower limit of 95% confidence interval for $p$:

$p_L$ such that

$$\Pr(X \geq n \mid p = p_L) = 0.025$$

$$\Rightarrow \Pr(X = n \mid p = p_L) = 0.025$$

$$\Rightarrow (p_L)^n = 0.025$$

$$\Rightarrow p_L = \sqrt[n]{0.025}$$

In the case $n = 25$ and $X = 25$, the 95% CI for $p$ is $(0.86, 1.00)$

Large $n$ and medium $p$

Suppose $X \sim \text{binomial}(n, p)$.

$$E(X) = np \quad \text{SD}(X) = \sqrt{n \cdot p(1 - p)}$$

$$\hat{p} = X/n \quad E(\hat{p}) = p \quad SD(\hat{p}) = \frac{\sqrt{p(1-p)}}{n}$$

For large $n$ and medium $p$, $\hat{p} \sim \text{normal}\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$

Use 95% confidence interval $\hat{p} \pm 1.96 \sqrt{\frac{p(1-p)}{n}}$

Unfortunately, this usually behaves poorly.

Fortunately, you can just use `binom.test()`