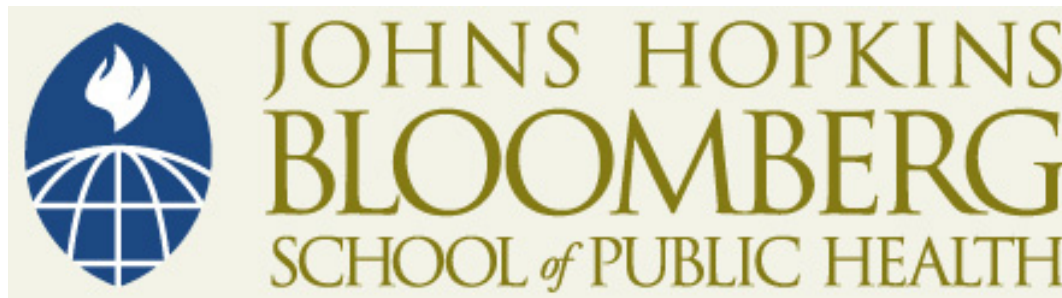


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# Population Change and Projection

**Stan Becker, PhD**  
**Bloomberg School of Public Health**

## Section A

*Rate of Change, Doubling Time, and  
the Relationship between Age  
Distinction and Demographic Rates*

# Population Change

- ◆ Book-keeping equation
- ◆ Let  $P_t$  = Population at time "t"  
 $P_0$  = Population at an earlier time "0"  
 $B$  = Births between time "0" and time "t"  
 $D$  = Deaths between time "0" and "t"  
 $I$  = In-migration / immigration between time "0" and "t"  
 $O$  = Out-migration / emigration between time "0" and "t"

*Continued*

# Population Change

- ◆ The population changes by adding births and in-migrants and subtracting deaths and out-migrants

$$P_t = P_0 + B - D + I - O$$

- ◆ Let  $b$  = Number added per time unit

$$b = \frac{P_t - P_0}{t}$$

# Linear Change

- ◆ The population size changes by exactly the same amount “b” during each time period “t”

$$P_t = P_0 + bt$$

# Linear Annual Rate of Change

## *Beginning of Period Approximation*

- ◆  $r_i$  = linear annual rate of change  
(beginning of period approximation)

$$r_i = \frac{P_t - P_0}{tP_0}$$

$$P_t = P_0 + r_i * t * P_0$$

# Linear Annual Rate of Change

*Arithmetic or Mid-Point Approximation*

- ◆  $r_m$  = linear annual rate of change  
(arithmetic or mid-point approximation)

$$r_m = \frac{b}{1/2(P_0 + P_t)} = \frac{P_t - P_0}{t/2(P_0 + P_t)}$$



# Exercise

## *Linear Annual Growth Rate*

- ◆ The Mexican population was estimated at 77,938,288 at the 1985 census and at 91,158,290 at the 1995 census
- ◆ Calculate the linear annual growth rate using both types of approximations

*You have 15 seconds to calculate the answer. You may pause the presentation if you need more time.*

# Exercise Answer

## *Linear Annual Growth Rate*

- ◆ The correct answers are as follows:

$r_i$   
**1.70%**

$r_m$   
**1.56%**

# Problem with the Linear-Change Approach

- ◆ Adding a fixed amount in each time period  
 $\Rightarrow$  the rate of change decreases because  
 $b / Pt \rightarrow 0$
- ◆ Say  $b = P_0$
- ◆ Then  $P_1 = P_0 + P_0 = 2P_0$   
 $P_2 = P_1 + P_0 = 3P_0$

# Problem with the Linear-Change Approach

◆ So

$$r_1 = \frac{P_1 - P_0}{1 * P_0} = 1$$

$$\begin{aligned} r_{m1} &= \frac{P_1 - P_0}{1/2(P_0 + P_1)} \\ &= \frac{P_0}{3/2 P_0} = \frac{2}{3} \end{aligned}$$

# Problem with the Linear-Change Approach

- ◆ And each continues declining

$$\begin{aligned} r_2 &= \frac{P_2 - P_1}{1 * P_1} \\ &= \frac{3P_0 - 2P_0}{2P_0} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} r_{m2} &= \frac{P_2 - P_1}{1/2(P_1 + P_2)} \\ &= \frac{P_0}{5/2P_0} = \frac{2}{5} \end{aligned}$$

# Geometric Change

- ◆ The population changes by step, i.e., the increment added (or the decrement is subtracted) periodically

$$P_t = P_0(1 + r_g)^t$$

- ◆ Where  $r_g$  = geometric rate of growth

# Geometric Annual Rate of Change

$$r_g = \sqrt[t]{\frac{P_t}{P_0}} - 1 = e^{\frac{1}{t} \ln\left(\frac{P_t}{P_0}\right)} - 1$$

# Exercise

## *Geometric Annual Rate of Change*

- ◆ The Mexican population was estimated at 77,938,288 at the 1985 census and at 91,158,290 at the 1995 census
- ◆ Calculate the geometric annual growth rate

*You have 15 seconds to calculate the answer. You may pause the presentation if you need more time.*



# Exercise Answer

## *Geometric Annual Rate of Change*

- ◆ The correct answer for  $r_g$  is as follows:
  - **1.58%**

# Exponential Change

- ◆ The population changes instantaneously and continuously

$$P_t = P_0 e^{rt}$$

# Exponential Growth Rate

- ◆ Exponential growth rate

$$r = \frac{1}{t} \ln \left( \frac{P_t}{P_0} \right)$$

- ◆ The advantage of the exponential method is that it's easy to solve for any parameter

$$t = \frac{\ln \left( \frac{P_t}{P_0} \right)}{r}$$

$$P_0 = \frac{P_t}{e^{rt}}$$

# Exercise

## *Exponential Growth Rate*

- ◆ The Mexican population was estimated at 77,938,288 at the 1985 census and at 91,158,290 at the 1995 census
- ◆ Calculate the exponential growth rate

*You have 15 seconds to calculate the answer. You may pause the presentation if you need more time.*

# Exercise Answer

## *Exponential Growth Rate*

- ◆ The correct answer for “r” is as follows:
  - **1.57%**

# Relation Between Rates of Change

## *Geometric and Exponential*

$$r_g = e^r - 1$$

$$\text{since } e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\text{then } r_g = r + \frac{r^2}{2!} + \dots$$

$$\Rightarrow r_g > r$$

# Rate of Natural Increase

- ◆ Let  $B$  = Births during a calendar year  
 $D$  = Deaths during same year  
 $P$  = Midyear population  
 $b$  = Birth rate  
 $d$  = Death rate

# Rate of Natural Increase

- ◆ Most direct indication of how rapidly a population actually grew as the result of vital processes

$$r = \frac{B - D}{P} * 1000$$
$$= b - d$$

$$r < 0 \Rightarrow D > B$$

$$r = 0 \Rightarrow B = D$$

$$r > 0 \Rightarrow B > D$$

- ◆ Is influenced by age structure



# Doubling Time

- ◆ Calculation based on exponential growth ("r" fixed in time)
- ◆ Allows determination of how long it will take a population to double in size (i.e., time for  $P_t / P_0 = 2$ )

# Doubling Time

- ◆ Since

$$t = \frac{\ln\left(\frac{P_t}{P_0}\right)}{r}$$

- ◆ Then for doubling time

$$t = \frac{\ln(2)}{r}$$

# Doubling Time

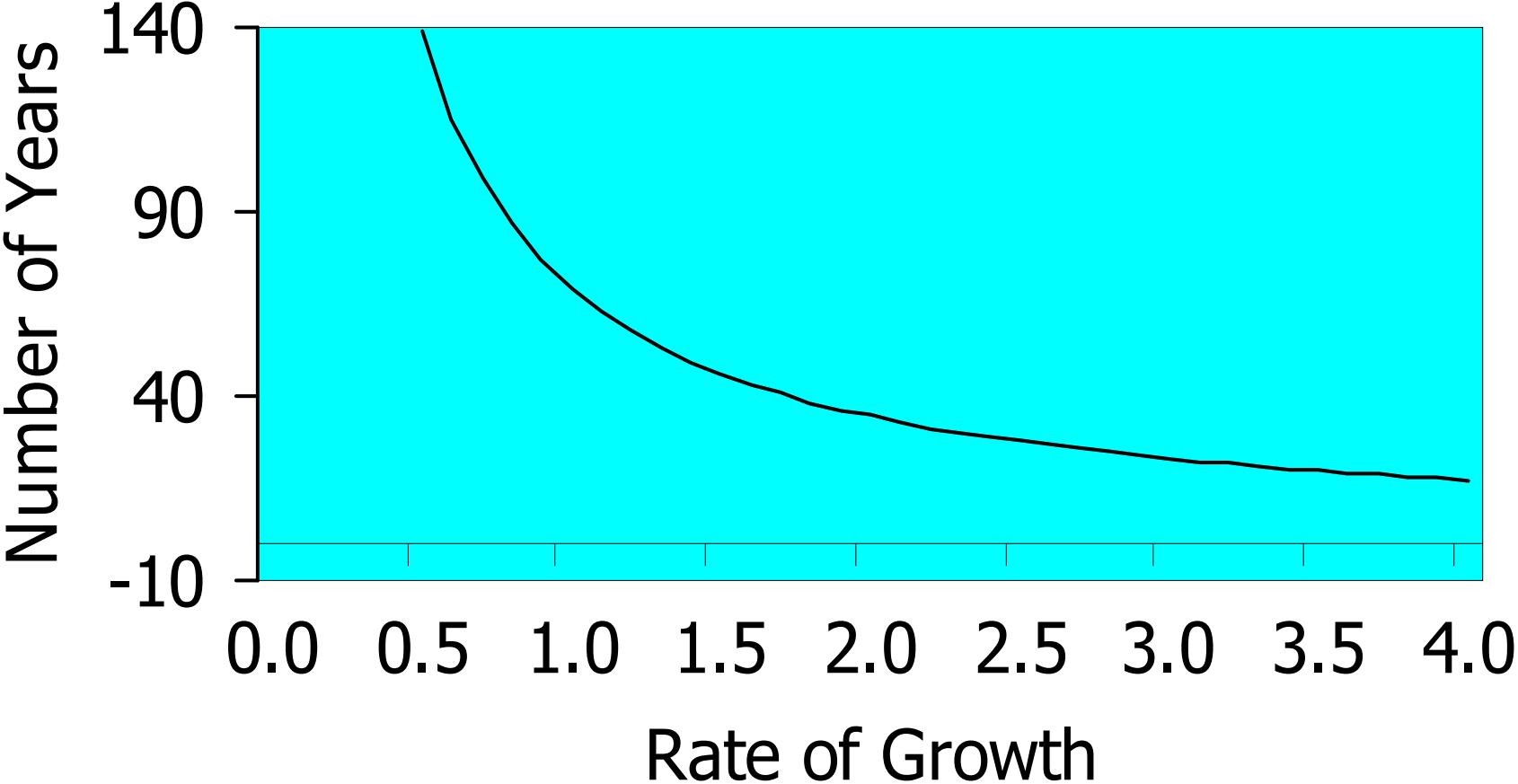
- ◆ Approximation

$$t = \frac{70}{r * 100}$$

# Some Examples of Doubling Times

Rate of Growth	Number of Years	Rate of Growth	Number of Years
1.0	69.3	2.0	34.7
1.1	63.0	2.1	33.0
1.2	57.8	2.5	27.7
1.3	53.3	3.0	23.1
1.4	49.5	3.5	19.8
1.5	46.2	4.0	17.3

# Number of Years Necessary for the Population to Double at Given Fixed Annual Rates of Growth



# Exercise

## *Doubling and Tripling Time*

- ◆ The Mexican population was estimated at 77,938,288 at the 1985 census and at 91,158,290 at the 1995 census
- ◆ Assuming Mexico's population is growing exponentially, what is its doubling time?
- ◆ In what year will the mid-1975 population have tripled?

*You have 15 seconds to calculate the answer. You may pause the presentation if you need more time.*

# Exercise Answer

## *Doubling and Tripling Time*

- ◆ The correct answers are as follows:
  - **Doubling time = 44.2 years**
  - **Tripling time = 70.1 years**
  - **Year will triple = 2045**

# The Relationship between Age Distribution and Demographic Rates

- ◆ Age distribution of a population gives a record of the demographic history of that population



# The Relationship between Age Distribution and Demographic Rates

- ◆ Age distribution is sensitive to changes in demographic rates, particularly fertility and, for small areas, migration rates
- ◆ In a closed population (i.e., no migration) the age distribution is determined only by fertility and mortality

# Summary

- ◆ Population change is measured by the difference between population sizes at different dates
- ◆ As time goes by, the population in an area can do the following:
  - Grow by adding births or people moving in the area (in-migrants)
  - Or it can decrease because of deaths or people moving out of the area (out-migrants)

# Summary

- ◆ The simplest method to estimate the population at a later time from that of an earlier date is the bookkeeping method, which adds the births and in-migrants to the initial population and subtracts the deaths and out-migrants

# Summary

- ◆ Mathematical models have been developed to estimate the average annual rate of change based on different assumptions on the way the population has grown during the period

# Section B

## *Projection*

# Population Projection

- ◆ *Population Projection*—Forecast of population change using estimates of fertility, mortality, and migration
- ◆ Projections may extend for varying numbers of years into the future
- ◆ Note:
  - Extrapolation = projection
  - Interpolation = estimation

# Population Projection

- ◆ Long-term projections (over 25 years) are used in connection with the development of natural resources, planning for provision of food, for transportation and recreational facilities, etc.
- ◆ Middle-range projections (10–25 years) are used for planning educational and medical facilities and services, housing needs, etc.

# Component Method

- ◆ Has become the standard methodology for projection
- ◆ Makes explicit the assumptions regarding the components of population growth—mortality, fertility, and net migration
- ◆ Gives insight into the way population changes



# Component Method

- ◆ Allows the user to estimate the effect of alternative levels of fertility, mortality, or migration on population growth
- ◆ Used to obtain projections of age-sex structure

# General Principles

- ◆ Start with the population distributed by age and sex at base date
- ◆ Apply assumed survival rates and age-sex specific fertility rates to obtain number of persons alive at the end of a unit of time

# General Principles

- ◆ Make allowance for net migration by age and sex, if desired
- ◆ Generally five-year age groups are used
- ◆ Projection interval must be integer multiple of age interval

# Population Projection

- ◆ Projection assuming closed population, constant fertility, and constant mortality
    - Let  $x = \text{Age } 5, 10, \dots, w$   
 $t = \text{Time "t"}$   
 ${}_5L_x = \text{Life table number of persons between ages "x" and "x+5" at time "t"}$
- $\frac{{}_5L_x}{{}_5L_{x-5}} = \text{Survival ratio from age group (x-5, x) to age group (x, x+5)}$

# Population Projection

$$P_{x, x+5}^{t+5} = P_{x-5, x}^t * \frac{{}_5L_x}{{}_5L_{x-5}}$$

- ◆ Projection of the population aged 55 to 59 from 1990 to 1995

$$P_{60-64}^{1995} = P_{55-59}^{1990} * \frac{{}_5L_{60}}{{}_5L_{55}}$$

# Treatment of Open-Ended Interval

$$P_{w+}^{t+5} = P_{w-5}^t * \frac{5L_w}{5L_{w-5}} + P_{w+}^t * \frac{T_{(w+5)}}{T_w}$$

- ◆ Where  $w+$  refers to the oldest age group

# Treatment of Open-Ended Interval

- ◆ Keyfitz's method

$$P_{85+}^{t+5} = {}_5P_{80}^t * \frac{{}_5L_{85}}{{}_5L_{80}} + P_{85+}^t * \frac{T_{90}}{T_{85}}$$

– In stationary population

$$\frac{{}_5P_{80}}{P_{85+}} = \frac{{}_5L_{80}}{T_{85}} \Rightarrow P_{85+} = \frac{{}_5P_{80} * T_{85}}{{}_5L_{80}}$$

# Treatment of Open-Ended Interval

## – Substituting

$$P_{85+}^{t+5} = {}_5P_{80}^t * \left( \frac{{}_5L_{85}}{{}_5L_{80}} + \frac{T_{85}}{{}_5L_{80}} * \frac{T_{90}}{T_{85}} \right)$$

$$= {}_5P_{80} * \left( \frac{{}_5L_{85} + T_{90}}{{}_5L_{80}} \right)$$

$$= {}_5P_{80} * \frac{T_{85}}{{}_5L_{80}}$$



# Getting $P_{0-4}$ at Time $t+5$

- ◆ Let  ${}_5L_x$  = Number of survivors between ages "x" and "x+5"
- ${}_5F_x^f$  = Female fertility rate for age group (x, x+5)
- x = Age "x"

# Getting $P_{0-4}$ at Time $t+5$

- ◆ Also let  $\alpha$  = Beginning age of reproduction  
 $\beta$  = End of reproductive age  
 ${}_5F_x$  = Female fertility rate for age group  $x, x+5$

# Getting $P_{0-4}$ at Time $t+5$

${}_5P_0^{t+5}$  = Survivors to time "t+5" of births that occurred between "t" and "t+5"

$$= \frac{{}_5L_0}{2l_0} * \left[ \sum_{x=a-5}^{\beta-5} \left( {}_5F_x^f + \frac{{}_5L_{x+5}}{{}_5L_x} * {}_5F_{x+5}^f \right) * {}_5P_x \right]$$

# Getting $P_{0-4}$ at Time $t+5$ - Derivation

- ◆ Births in one year =

$$\left( \frac{1}{2} \right) * \left( {}_5P_{15} + \frac{{}_5L_{15}}{{}_5L_{10}} * {}_5P_{10} \right) * {}_5F_{15} +$$

$$\dots + \left( \frac{1}{2} \right) * \left( {}_5P_{40} + \frac{{}_5L_{40}}{{}_5L_{35}} * {}_5P_{35} \right) * {}_5F_{40}$$

$$= \left( \frac{1}{2} \right) * \sum_{x=a-5}^{\beta-5} \left( {}_5F_x + \frac{{}_5L_{x+5}}{{}_5L_x} * {}_5F_{x+5} \right) * {}_5P_x$$

*Continued*

# Getting $P_{0-4}$ at Time $t+5$ - Derivation

- ◆ Births in five years =

$$5 * \left[ (1/2) * \sum_{x=a-5}^{\beta-5} \left( {}_5F_x + \frac{{}_5L_{x+5}}{{}_5L_x} * {}_5F_{x+5} \right) * {}_5P_x \right]$$

# Population Projection

- ◆ To get  $P_{0-4}$ , births must survive an average of 2.5 years

$${}_5P_0^{t+5} = \frac{5}{2} * \frac{{}_5L_0}{5l_0} * \left[ \sum_{x=a-5}^{\beta-5} \left( {}_5F_x + \frac{{}_5L_{x+5}}{{}_5L_x} * {}_5F_{x+5} \right) * {}_5P_x \right]$$

$$= \frac{{}_5L_0}{2l_0} * \left[ \sum_{x=a-5}^{\beta-5} \left( {}_5F_x + \frac{{}_5L_{x+5}}{{}_5L_x} * {}_5F_{x+5} \right) * {}_5P_x \right]$$

$$= \frac{{}_5L_0}{2l_0} * \left[ \sum_{x=a-5}^{\beta-5} \left( {}_5P_x + \frac{{}_5L_{x+5}}{{}_5L_x} * {}_5P_{x+5} \right) * {}_5F_x \right]$$

# Matrix Representation

- ◆ 1st row represents fertility rates and 0–5 survival
- ◆ Sub-diagonal represents survival ratios from one age group to another
- ◆ Note: The rest of the matrix (L) has zeros

# Matrix Representation

$$\mathbf{L} = \begin{bmatrix}
 0 & 0 & \frac{L_0}{2l_0} \left( \frac{L_{15}}{L_{10}} * F_{15} \right) & \frac{L_0}{2l_0} \left( F_{15} + \frac{L_{20}}{L_{15}} * F_{20} \right) & \dots & 0 & 0 \\
 \frac{L_5}{L_0} & 0 & 0 & 0 & \dots & 0 & 0 \\
 0 & \frac{L_{10}}{L_5} & 0 & 0 & \dots & 0 & 0 \\
 \cdot & \cdot & & \cdot & & \cdot & \cdot \\
 \cdot & \cdot & & \cdot & & \cdot & \cdot \\
 \cdot & \cdot & & \cdot & & \cdot & \cdot \\
 0 & 0 & 0 & 0 & \dots & \frac{L_{w-5}}{L_{w-10}} & \frac{T_w}{T_{w-5}}
 \end{bmatrix}$$



# Population Projection

- ◆ To get the population at a later date

$$L * P^t = P^{t+5}$$

$$L * P^{t+5} = P^{t+10} = L^2 * P^t$$

etc.

# Population Projection

- ◆ To project the total population at a later date
  - First, use the female projection and fertility rates for both boy and girl births
  - Then, put the male births into male projection
  - Note: The male matrix only contains survival ratios

# Projection Adjusting for Net Migration (NM)

$$P^{t+5} = L * P^t + NM$$

- ◆ Where NM = Net migration
- ◆ It is easiest to take net migration into account at the end of each time interval
- ◆ Getting estimates of net migration by age and sex, and projected numbers by age and sex, may be difficult

# Projection Adjusting for Changes in Fertility and Mortality

- ◆ Let  $L^i$  = Projection matrix for  $i^{\text{th}}$  interval  
 $L^0$  = Projection matrix for first interval

# Population Projection

- ◆ Ex: At time  $t+10$

$$P^{t+10} = L^1 * (L^0 * P^t)$$

- ◆ Or more generally

$$P^{t+5*n} = L^{n-1} * L^{n-2} \dots * L^0 * P^t$$

- ◆ Where  $L^i$  = Projection matrix for  $i^{\text{th}}$  time interval

# Population Projection

- ◆ Note: Stable population results from unchanging rates of births and deaths
- ◆ Since  $L_n$  “stabilizes” when “ $n$ ” gets large (i.e., for every element), its value in “ $L^{n+1}$ ” divided by its value in “ $L^n$ ” converges to a constant “ $\lambda$ ”

# Population Projection

- ◆ Therefore:

$$L^{n+1} * P^t = \lambda * L^n * P^t$$

$$\text{so } \lambda * P^t = P^{t+5}, \text{ and}$$

$$\lambda = \frac{P^{t+5}}{P^t} = e^{5r} \text{ in continuous notation}$$

# Population Projection

- ◆ Several variations of projection exist for changing rates
  - Change fertility rates by  $x\%$  and/or mortality rates by  $y\%$ , at each step
  - Fix final levels of fertility and mortality and interpolate between the two time periods
  - Note: Net migration can also be easily made to vary



# Projection for Small Areas

- ◆ Internal migration (e.g., rural-urban migration) may be important
- ◆ Population projection for political units with relatively few boundary changes (e.g., states) and for statistical subdivisions with boundaries changing (e.g., urban/rural) are two different problems

# Ratio-Correlation or Regression Methods

- ◆ Define:

$$Y_i = \frac{\text{percent of total population n area i, time 1}}{\text{percent of total population n area i, time 0}}$$

- ◆ Write equation (can use different number of Xs):

$$Y_i = \alpha + \beta_1 X_{i1} + \beta_2 X_{i2} + \dots$$

# Projection for Small Areas

- ◆ For example:

$$X_{i1} = \frac{\text{percent of total births area } i, \text{ time } 1}{\text{percent of total births area } i, \text{ time } 0}$$

$$X_{i2} = \frac{\text{percent of total school enrollees area } i, \text{ time } 1}{\text{percent of total school enrollees area } i, \text{ time } 0}$$

# Projection for Small Areas

- ◆ Estimate  $\alpha$ ,  $\beta_1$ ,  $\beta_2$  statistically, and use these to predict  $Y_i$  for some future time  $t_2$ 
  - Note:  $Y_i$  is a percent and if all areas are done, the percentages may not add to 100
- ◆ So, they must be adjusted to 100 and then multiplied by the estimated total population at  $t_2$  to get the projection for area “i” at that time

# Administrative Records

$$P_t = H_t \times PPH_t + GQ_t$$

- ◆  $P_t$  = Total population at time "t"
- $H_t$  = Occupied housing units at time "t"
- $PPH_t$  = Average number of persons per household at time "t"
- $GQ_t$  = Population in group quarters at time "t"

# Combined Methods

- ◆ Average of several methods
- ◆ Method used by the U.S. Census Bureau

# Judging Projections

- ◆ The evaluation of projections requires some standard by which to judge their quality
- ◆ One may reasonably compare a projection with the population actually recorded later using percent differences to indicate how far it deviates from the actual figure

# Judging Projections

- ◆ The concept of “accuracy” becomes less meaningful where several series of projections are offered as reasonable possibilities, and particularly where none is offered as a “forecast”
- ◆ The United Nations uses high, medium, low, and constant (unchanging fertility and mortality) projections



# Judging Projections

- ◆ Where possible, it may be more profitable to compare the actual components of population change (i.e., births, deaths, and net migration) with their projected figures because it provides insight into the reasonableness of the various assumptions

# Proportional Error

- ◆ Hindsight, calculate the proportional error:

$$\text{Proportional error} = \frac{P_{\text{actual}} - P_{\text{proj}}}{P_{\text{actual}}}$$

- ◆ Where  $P_{\text{actual}}$  = Actual population  
(e.g., census count at  
time "t")  
 $P_{\text{proj}}$  = Population projection  
for time "t"

# Population Projection

- ◆ One can also assess projections by looking at the width of the range from the highest and the lowest series in a set of principal projections

# Population Projection

- ◆ This width of the range depends on the regularity of the following:
  - Past demographic trends, knowledge regarding past trends
  - Ability to measure them accurately
  - Analyst's judgment of the likely course of future change

# Population Projection

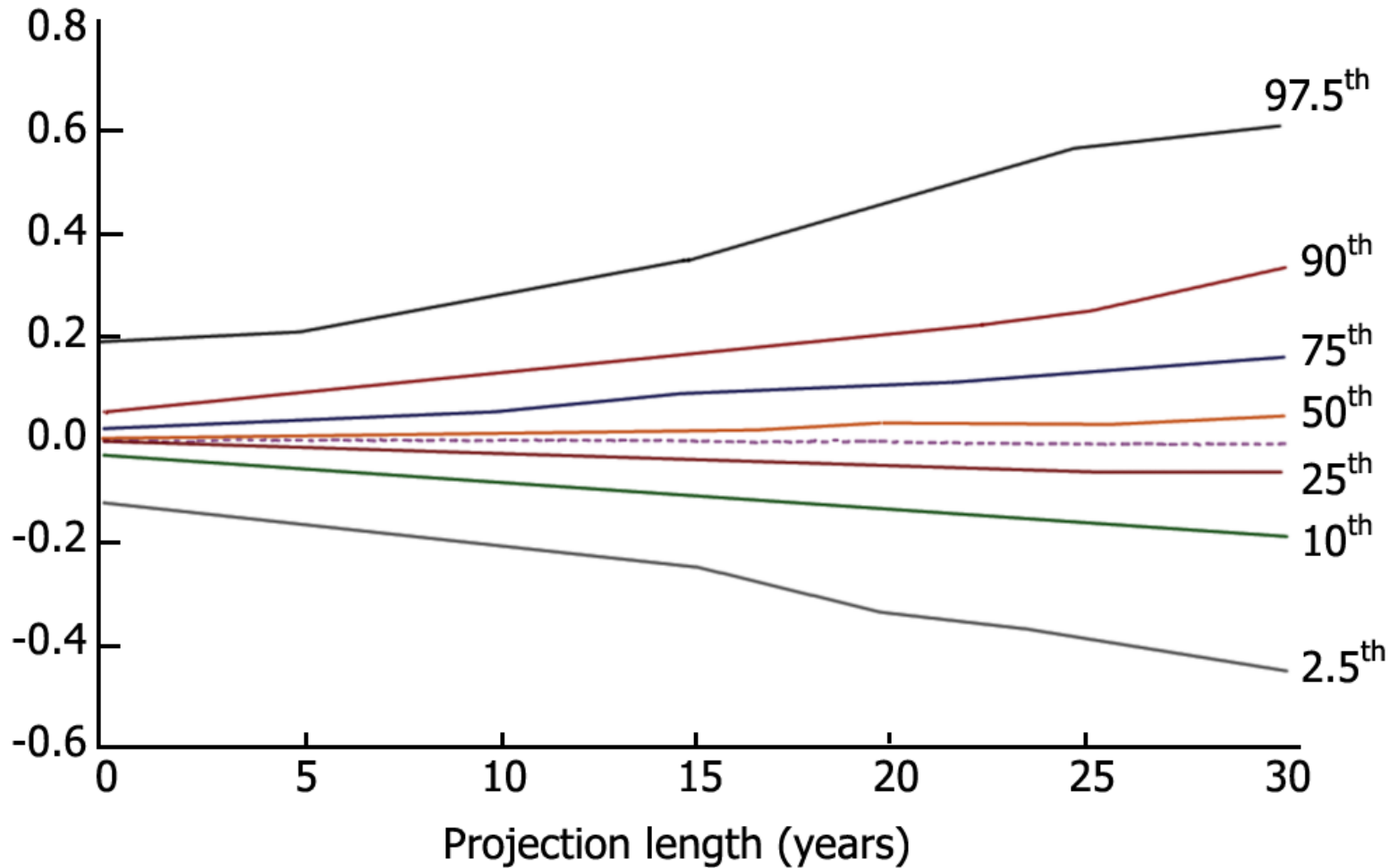
- ◆ As the range widens, the analyst is indicating that he/she has less and less confidence in the projections

# Error in Projected Population by Target Year: Means across Countries and Forecasts

<b>Target year</b>	<b>Proportional error</b>	<b>Absolute proportional error</b>	<b>(n)</b>
1975	-0.004	0.062	163
1980	0.009	0.076	439
1985	0.010	0.069	635
1990	0.016	0.080	1,194
1995	0.025	0.089	1,138
2000	0.036	0.109	1,375

Adapted from John Bongaarts and Rodolfo A. Bulatao, Editors, Beyond six billion, National Academy Press, 2000

# Proportional Error by Projection Length: Percentile for All Countries



Source: John Bongaarts and Rodolfo A. Bulatao, Editors, *Beyond six billion*, National Academy Press, 2000

# Uses of Projections

- ◆ Forecasting future population change
- ◆ Warning of population increase or decrease
- ◆ Sensitivity analyses to see how population will change with changes in assumptions about fertility, mortality, and/or migration



# Examples of Applications

- ◆ Projection of population by age and sex to estimate the age distribution of the population at a later date
- ◆ Projection of children of school-age and apply school enrollment ratios to evaluate the needs in teachers and schools
- ◆ Projection for the labor force

# Computer Programs

- ◆ Examples:
  - DEMPROJ (Futures Group International) to create projections for policy presentations or planning exercises and to produce the inputs required by the other programs of the integrated package SPECTRUM
  - IPSS (SENECIO Software Inc.) for interactively exploring population dynamics, population projection, and the life table program for Macintosh

# Population Projection

- ◆ Contact:

Director

Population Division, United Nations  
United Nations Plaza, Rm. DC2-1950  
New York, NY 10017 USA

Fax: (212) 963-2147

Email: [population@un.org](mailto:population@un.org)

Population Information Network (POPIN) web site:  
<http://www.undp.org/popin>

# Summary

- ◆ Population projection is a forecast of population change using estimates of fertility, mortality, and migration
- ◆ No projection is 100 percent accurate, i.e., there is always some degree of uncertainty
- ◆ The United Nations uses low, medium, high, and constant projections

# Summary

- ◆ Various computer programs have been developed to help the demographer simulate the effects on population size of different rates of fertility and mortality